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LETTER TO THE EDITOR

Shortest paths in percolation

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Abstract. The separation of two points on a percolation network is characterised not only by the distance between them, but also by the length of a path on the network which connects them. The wetting velocity v provides a measure of the lengths of the shortest connecting paths on the network above the percolation concentration p_c. Along the easy axes, v is expected to vanish as $(p-p_c)^\theta$ near p_c, while $(1-v)$ is expected to vary as $(p_d-p)^{\theta'}$ near the directed percolation concentration p_d. The exponent θ is related to an exponent $\tilde{\nu}$ which characterises the shortest paths precisely at p_c, and which has recently been determined numerically. The wetting velocity is calculated analytically on a randomly diluted Bethe lattice, and the values $\theta=\frac{1}{2}$ and $\theta'=1$ (with logarithmic corrections) are found. The direction dependence of v is also investigated.

The percolation problem (Broadbent and Hammersley 1957) is concerned with the nature of connections in a lattice in which a randomly chosen subset of elements (sites or bonds) has been removed. For instance, in site percolation, one asks for the probability that two sites i and j are connected by at least one unbroken chain of sites. But besides asking whether connections exist, it is also of interest to ask how tortuous they are. One is thus led to consider the shortest (possibly non-unique) path between site i and site j—one on which a walker must traverse the fewest possible sites in order to go from i to j, if these sites are connected in a particular realisation of the lattice. There are several ways of looking at the problem. The length l_{ij} of the shortest path is equal to the time taken by a fluid injected at site i to reach site j, assuming that the fluid front moves outward and wets neighbouring accessible sites every second. The motion of the fluid front defines the wetting velocity v (Dhar 1982) whose value in a particular direction provides a measure of the shortest path length in that direction. The problem is an example of first passage percolation (Hammersley and Welsh 1965), with a particular distribution of transmission times. The shortest path length l_{ij} is sometimes also referred to as the chemical distance between sites i and j, and its behaviour exactly at the critical percolation concentration has been studied by series and Monte Carlo methods (Alexandrowicz 1980, Pike and Stanley 1981, Hong and Stanley 1983, Puech and Rammal 1983, Herrmann *et al* 1984, Havlin and Nossal 1984).

Given two sites on a hypercubic lattice separated by a distance R_{ij}, the wetting velocity v is defined as

$$v = R_{ij}/l_{ij} \tag{1}$$

where l_{ij} is the length of the shortest connecting path between i and j. We take R_{ij} to be the sum of the Cartesian separations $|x_i-x_j|+|y_i-y_j|+\ldots$, rather than the Euclidean distance between i and j, so that $R_{ij}\leq l_{ij}$, with the equality holding for the smallest possible chemical separation between two fixed sites i and j. The value of v ranges from zero (which characterises extremely tortuous paths) to one (for the shortest, most direct paths). In general, v is quite anisotropic (Dhar 1982). It is largest along the easy directions (any of the body diagonals of the hypercubic lattice), along which its value varies continuously from 0 to 1 as p increases from the percolation concentration p_c to the directed percolation concentration p_d. Of particular interest is the singular fashion in which the limits $v=0$ and $v=1$ are reached. We expect that along the easy direction

$$v \sim (p-p_c)^{\theta} \qquad \text{as } p \to p_c, \tag{2}$$

$$1-v \sim (p_d-p)^{\theta'} \qquad \text{as } p \to p_d, \tag{3}$$

would hold. Calculations on the Bethe lattice (described below) support this expectation with $\theta=\frac{1}{2}$ and $\theta'=1$ with logarithmic corrections, but before we turn to these, we give an argument which relates θ to an exponent $\tilde{\nu}$ which governs the relationship between l_{ij} and R_{ij} exactly at p_c (Havlin and Nossal 1984):

$$\langle R_{ij}^2 \rangle \sim l_{ij}^{2\tilde{\nu}}. \tag{4}$$

In order to relate (2) and (4) above, consider the infinite percolation cluster with p just above p_c. On length scales much smaller than the correlation length ξ, its structure is expected to be similar to that of the infinite cluster at p_c, whereas it is expected to be homogeneous on length scales larger than ξ. Consistent with this picture, one expects the shortest path length between two sites a distance $n\xi$ apart ($n \gg 1$) to be roughly n times the shortest path length $\Lambda(\xi)$ between two sites a distance ξ apart. But $\Lambda(\xi)$ is given, on using equation (4), by $\Lambda(\xi) \sim \xi^{1/\tilde{\nu}}$ and ξ itself diverges as $(p-p_c)^{-\nu}$. The wetting velocity v, which measures the ratio of the separation to the shortest path length, thus follows equation (2) with

$$\theta = \nu(1/\tilde{\nu}-1). \tag{5}$$

This relationship can be checked on the Bethe lattice, on which $\nu=\frac{1}{2}$. The value $\theta=\frac{1}{2}$ we find below is consistent with $\tilde{\nu}=\frac{1}{2}$ (Havlin and Nossal 1984).

We now turn to the determination of the shortest paths in percolation on the Bethe lattice. The latter is part of an endlessly branching structure with no loops. The customary percolation problem has been solved earlier on this pseudo-lattice (Fisher and Essam 1961), as have the related conductivity and diffusion problems (Stinchcombe 1974, Straley 1980). The solutions obtained have proved to be quite instructive, and the critical exponents obtained thereby provided reference points for expansions around the upper critical dimension (Toulouse 1974). However, before considering the effects of random dilution, let us define separations between sites on the Bethe lattice.

Consider an undiluted Bethe lattice with coordination number $2m$, where m is an integer. Each link is assigned a type and an arrow. There are m types $\alpha_1, \alpha_2 \ldots \alpha_m$ (the type is like a coordinate axis label; see below). At every site, there are two links of type α_k ($k=1,\ldots m$), one with an arrow pointing into the site and the other with the arrow pointing away from it. A geometrical interpretation of the assignment of types and arrows is as follows. Imagine the Bethe lattice embedded in an infinite-dimensional space which is spanned by unit vectors $\hat{e}_1, \hat{e}_2 \ldots \hat{e}_m, \hat{e}_{m+1} \ldots \hat{e}_\infty$. A link of type α_k ($1 \leq k \leq m$) is one which has a component ± 1 along $\hat{e}_k$, but vanishing components along the $(m-1)$ directions $\hat{e}_l$ ($l=1\ldots m$; $l \neq k$). The arrow denotes the positive sense along the kth axis. The components along the remaining directions from $(m+1)$ to ∞ are chosen so that every link is orthogonal to every other, but these components need not be specified as we will confine our attention to the m-dimensional subspace spanned by $\hat{e}_1, \hat{e}_2 \ldots \hat{e}_m$. For k between 1 and m, the kth component of the displacement between sites i and j is

$$r_{ij}^k = \sum_n s_n^k \tag{6}$$

where the sum runs over all steps of the unique self-avoiding walk (SAW) from site i to j, and s_n^k is $1(-1)$ if the nth step is along (against) the direction of the arrow on a link of type α_k, while it vanishes if the link is not of type k. It suffices to confine our attention to the 'positive' sector, in which every displacement r_{ij}^k is non-negative, as the behaviour we find in this sector occurs, by symmetry, in each of the other $(2m-1)$ sectors as well. In the positive sector, the sum of displacements r_{ij}^k serves as a measure of the separation R_{ij} of two points. We can write R_{ij} in terms of pseudospins $s_n = \pm 1$ associated with links (we will make contact with an Ising model below; see also White and Barma (1984)). Take $s_n = 1(-1)$ if the nth link in the SAW from i to j is traversed along (against) the arrow on that link. Then we have

$$R_{ij} = \sum_n s_n \tag{7}$$

while the length of the connecting path between i and j

$$l_{ij} = \sum_n |s_n| = N \tag{8}$$

is just the number of steps of the SAW connecting the two points.

Now consider randomly diluting the Bethe lattice by removing a fraction $(1-p)$ of the sites. It is known (Fisher and Essam 1961) that if p exceeds a critical value

$$p_c = 1/(2m-1) \tag{9}$$

then there is a finite probability that there are connected paths leading from a given site 0 to infinity. We can see this as follows. The end point of each N-step SAW with origin 0 and executed on the undiluted Bethe lattice identifies a distinct point on the lattice; the number of such points is $(2m-1)^N$. On the diluted lattice, the probability that the end point of each SAW is connected to 0 is p^{N+1}. Hence the expectation value C_N of the number of end points reached by N-step SAWs on the diluted lattice is $p[p(2m-1)]^N$. We see that C_∞ vanishes for $p<p_c$ whereas it diverges for $p>p_c$ on the Bethe lattice. Similarly, the expectation value D_N of end points reached through directed SAWs (in which every link is traversed along the arrow) is $p(mp)^N$. Recall that each directed SAW is characterised by $v=1$. We see that D_∞ vanishes if $p<p_d$ with

$$p_d = 1/m \tag{10}$$

whereas it diverges for $p>p_d$. In equations (9) and (10), p_c and p_d are, respectively, the critical percolation concentration and the critical directed percolation concentration on the Bethe lattice.

Just as p_d is the threshold concentration for the occurrence of an infinite path with $v=1$, we can ask for the corresponding threshold $p(v)$ for the occurrence of infinite paths with velocity v. Following the arguments in the previous paragraph, the expected number of N-step SAWs characterised by a velocity v is

$$C_N(v) = p^N Q_N(v) \tag{11}$$

where $Q_N(v)$ is the number of N-step SAWs on the undiluted lattice which lead to a displacement $R_{ij} = vN$. It is given by

$$Q_N(v) = \sum_{s_1 \ldots s_N} [T(s_1, s_2)T(s_2, s_3) \ldots T(s_{N-1}, s_N)]\delta\left(vN - \sum_{n=1}^{N} s_n\right) \tag{12}$$

where $T(s_n, s_{n+1})$ is the number of alternatives available at the $(n+1)$th step of the SAW, for specified values of s_n and s_{n+1}. Since the nth step blocks one of the options for the $(n+1)$th step, we have

$$\begin{aligned} T(s_n, s_{n+1}) &= m && \text{if } s_{n+1} = s_n, \\ &= m-1 && \text{if } s_{n+1} = -s_n. \end{aligned} \tag{13}$$

Identifying $T(s_n, s_{n+1})$ with $\exp(Ks_ns_{n+1})$, we see that $Q_N(v)$ is the partition function of a 1D Ising model in an ensemble in which the magnetisation v is held fixed. The corresponding free energy is

$$a(v) = N^{-1} \ln Q_N(v) \tag{14}$$

but for calculational purposes it is more convenient to specify the magnetic field rather than to fix v. The resulting free energy

$$f(h) = N^{-1} \ln\left(\sum_{v} Q_N(v)\, \mathrm{e}^{Nvh}\right) \tag{15}$$

is related to $a(v)$ by a Legendre transform if N is large. We have

$$a(v) = f(h) - vh \tag{16}$$

with

$$v = \partial f/\partial h. \tag{17}$$

The calculation of $f(h)$ is carried out straightforwardly on introducing an h-dependent transfer matrix (see equation (A8) of White and Barma (1984)). Its largest eigenvalue

$$\lambda(h) = m \cosh h + (m^2 \cosh^2 h - 2m + 1)^{1/2} \tag{18}$$

determines $f(h)$ for large N,

$$f(h) = \ln \lambda(h). \tag{19}$$

The condition for criticality is that $C_N(v)$ should neither vanish nor diverge as $N \to \infty$. Using (11), (14), (16) and (19), the condition can be expressed as

$$p = \mathrm{e}^{hv}/\lambda(h). \tag{20}$$

Using equation (17) we see that the critical locus is found by eliminating h between (20) and

$$v = \frac{m \sinh h}{(m^2 \cosh^2 h - 2m + 1)^{1/2}}. \tag{21}$$

In the limit $h \to \infty$, we have $v = 1$, and, as expected, the critical value of p is p_{d}. On the other hand, when $h \to 0$, we see that $v \to 0$, and the corresponding critical value of p is the undirected percolation concentration p_{c}. The wetting velocity as a function of p is shown by the bold curve in figure 1. Consider the approach to $v = 0$ and $v = 1$.

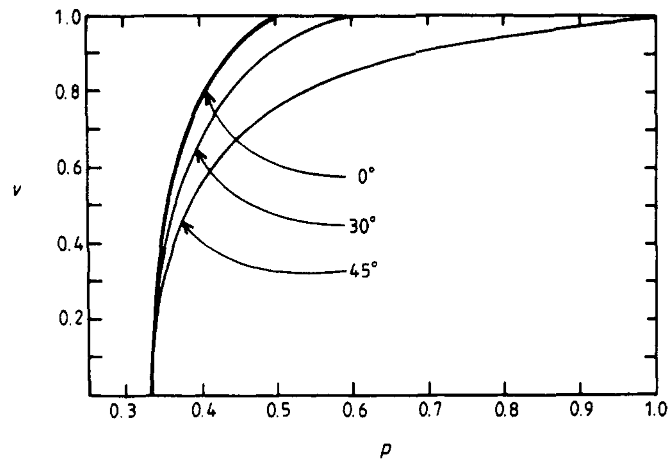

Figure 1. The wetting velocity $v(\phi)$ is plotted as a function of the site occupation probability p on a Bethe lattice with coordination number $4(m=2)$. Each curve corresponds to a different direction, and is labelled by the angle ϕ between that direction and the easy axis ($\phi=0°$).

As the critical concentration p_c is approached from above, the wetting velocity vanishes as

$$v \approx \left(\frac{2m}{m-1}\right)^{1/2}\left(\frac{p-p_c}{p_c}\right)^{1/2}, \qquad p \to p_c. \tag{22}$$

On the other hand, near the directed percolation threshold, we find

$$1-v \approx 2\delta p/[-\ln(2\delta p)], \qquad \delta p \to 0, \tag{23}$$

with $\delta p \equiv (p_d - p)/p_d$. From (22) and (23) we read off $\theta = \frac{1}{2}$ and $\theta' = 1$ (with logarithmic corrections) for the Bethe lattice.

In the discussion above, we did not specify the components of the separation R_{ij}. However, since we calculated the smallest value of p beyond which paths with a certain velocity v proliferate, we would expect the critical paths to be oriented predominantly along the m-dimensional body diagonal $(11\ldots)$, as such paths are most numerous. We now turn to an investigation of the critical value of p if we insist not only on a particular value of v, but also that the paths be oriented in a direction other than the body diagonal. We will study this feature on a Bethe lattice with $m=2$. To this end, let us first ask for the number $Q_N(v_1, v_2)$ of N-step SAWs on the undiluted Bethe lattice which lead to a displacement with components $\frac{1}{2}N(v_1+v_2)$ and $\frac{1}{2}N(v_1-v_2)$ along the two Cartesian axes. Without loss of generality, we require that $v_1 \geqslant v_2 \geqslant 0$. The displacement is characterised by an angle ϕ measured from the (11) direction, where ϕ is given by

$$\tan\phi = v_2/v_1. \tag{24}$$

It is expedient to introduce fields h_1 and h_2 and to calculate

$$\exp[Nf(h_1, h_2)] \equiv \sum_{v_1, v_2} Q_N(v_1, v_2)\exp[N(v_1 h_1 + v_2 h_2)] \tag{25}$$

rather than $Q_N(v_1, v_2)$ directly. The sum in (25) can be evaluated on rewriting Nv_1

and Nv_2 as sums over link variables and then introducing a transfer matrix. Explicitly, we have

$$Nv_1 = \sum_n s_n, \qquad Nv_2 = \sum_n t_n, \tag{26}$$

where $s_n = 1(-1)$ if the nth step of the SAW has a positive (negative) component along the 11 direction, while $t_n = 1(-1)$ if the nth step has a positive (negative) component along the $1\bar{1}$ direction. We can rewrite (25) as

$$\exp[Nf(h_1, h_2)] = \sum_{\{s_n, t_n\}} \prod_{n-1}^{N-1} \langle s_n t_n | \tilde{T} | s_{n+1} t_{n+1} \rangle. \tag{27}$$

The transfer matrix $\tilde{T}$ is given by

$$\begin{pmatrix} xy & 0 & x & y \\ 0 & x^{-1}y^{-1} & y^{-1} & x^{-1} \\ x & y^{-1} & xy^{-1} & 0 \\ y & x^{-1} & 0 & x^{-1}y \end{pmatrix} \tag{28}$$

with

$$x = \exp(h_1), \qquad y = \exp(h_2). \tag{29}$$

The largest eigenvalue of $\tilde{T}$ is

$$\lambda(h_1, h_2) = 2\cosh h_1 \cosh h_2 + (4\cosh^2 h_1 \cosh^2 h_2 - 3)^{1/2} \tag{30}$$

and in analogy to equation (20), the equation of the critical curve on the diluted lattice is given by

$$p = \exp(h_1 v_1 + h_2 v_2)/\lambda(h_1, h_2) \tag{31}$$

with

$$v_1 = \partial \ln \lambda(h_1, h_2)/\partial h_1 = \frac{2\sinh h_1 \cosh h_2}{(4\cosh^2 h_1 \cosh^2 h_2 - 3)^{1/2}} \tag{32}$$

and v_2 given by a similar equation with h_1 and h_2 interchanged. Equations (24), (31) and (32) determine the wetting velocity $v(\phi) \equiv v_1$ as a function of ϕ and p, on eliminating h_1 and h_2.

Let us consider some limits. When $h_2 = 0$, we have $\phi = 0$, which corresponds to the 11 direction. In this case we recover, as expected, the earlier results (18)-(21), with m set equal to 2 in those equations. The limit $h_1 = h_2$ corresponds to $\phi = 45°$, the orientation of the coordinate axis 10. It is hardest to form connections in this direction, and a value $v(45°) = 1$ for the wetting velocity is achieved only in the limit of the undiluted lattice, $p = 1$. Figure 1 shows the wetting velocity $v(\phi)$ as a function of p for $\phi = 0°$, 30° and 45°. The intercept of a curve (labelled by a particular ϕ) with the $v = 1$ axis picks out the smallest value $p = p_{\text{int}}$ beyond which infinitely long directed paths occur in that direction; thus the intercepts follow the dependence of the directed percolation cone angle on p (Dhar 1982). For small ϕ, we find

$$p_{\text{int}} - \tfrac{1}{2} \approx \tfrac{1}{4}\phi^2. \tag{33}$$

For fixed ϕ, the approach of the curve to the limits $v = 0$ and $v = 1$ is similar to the approach described by equations (22) and (23). In particular, the exponents $\theta = \frac{1}{2}$ and $\theta' = 1$ (with log corrections) do not change. The ϕ independence of θ is not really surprising as at, and close to, the percolation threshold, we would expect the wetting velocity to be isotropic, provided the Euclidean definition of distance is used. Our definition of v in terms of the Cartesian separation introduces a ϕ-dependent amplitude, but leaves the exponent θ unaltered.

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Note added in proof. After this letter was submitted, I learnt of related work by Ritzenberg and Cohen (1984). These authors propose a scaling for m for the behaviour of the shortest path length and give an argument not unlike that presented here, leading to the relation (5) between critical exponents.

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